FP - INJECTIVE MODULES AND RELATIVELY DIVISIBLE MODULES

Thesis submitted in partial fulfilment of the requirements
for the degree of Master of Science

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C O N T E N T S

ABSTRACT

An exact sequence of left modules $0 \quad A' \quad A \quad A'' \quad 0$ is called a pure sequence if for every right module D the sequence $0 \quad D \otimes_R A' \quad D \otimes_R A$ is exact. The sequence is called RD-pure if the condition holds for the modules D of the form $D = R/rR$.

R.B. Warfield had shown that over a Prüfer ring the two definitions are equivalent. We show that the converse is also true: If R is a commutative integral domain such that every RD-pure sequence is pure, then R is a Prüfer ring. If we do not insist on the assumption that the ring is an integral domain we get a similar equivalent condition: A commutative ring is a semi-hereditary ring iff for every RD-pure sequence $0 \quad A' \quad A \quad A'' \quad 0$ and for every ideal I in R the sequence $0 \quad R/I \otimes_R A' \quad R/I \otimes_R A$ is exact.

Another result of the same type is the following: A commutative integral domain is a Dedekind ring iff for every RD-pure sequence $0 \quad A' \quad A \quad A'' \quad 0$ and for every finitely presented module M, the sequence $\mathrm{Hom}_R(M,A) \quad \mathrm{Hom}_R(M,A'') \quad 0$ is exact.

An R-Module which is pure in every module containing it is called an FP-injective module. We show that the ring is a left semi-hereditary ring iff the left FP-injective dimension of the ring is less or equal to 1. Thus the ring is a left semi-hereditary ring iff it is a left coherent ring and its global weak dimension is not greater than 1. A corollary is that a ring which is a left

semi-hereditary ring and also a right coherent ring must also be a right semi-hereditary ring.

We introduce a definition of a "relatively divisible" module for a general ring, which coincides with the definition of a divisible module in case that the ring is an integral domain.

A quotient of a relatively divisible module is a relatively divisible module iff the ring is a PP-ring.

It is known that a commutative integral domain is a Dedekind ring iff every divisible module is injective. We get similar results: A commutative integral domain is a Prüfer ring iff every divisible module is FP-injective. We get also similar results in case that the ring is not an integral domain: A commutative ring is a semi-hereditary ring iff it is a PP ring such that every relatively divisible module is FP-injective. A commutative ring is a Noetherian hereditary ring iff it is a PP ring such that every divisible module is injective.

E. R. Gentile had proved that a left semi-hereditary ring which is FP-injective as a left module over itself must be a (von-Neumann) regular ring. We generalize his result: A left PP ring which is relatively divisible as a left module over itself must be a regular ring.

It is known that an integral domain over which every injective module is projective must be a division ring. We get a stronger result: An integral domain over which every injective module is flat must be a division ring.

The global weak dimension of the ring of matrices of a given rank over R is equal to the global weak dimension of the ring R.

The left FP-injective dimension of the ring of matrices over R is equal to the left FP-injective dimension of R. Therefore, the ring of matrices over a semi-hereditary ring is also a semi-hereditary ring.

If for every number n the ring of matrices R_n is a PP ring, then R is a left semi-hereditary ring.

If R is a commutative ring, M is a maximal ideal in R and A is an FP-injective module over R then A_m need not be FP-injective as an R_m-module.

SUMMARY

The fundamental theorem in this thesis is a reformulation of theorems of D. J. Fieldhouse and of P. M. Cohn characterizing pure sequences, the first of which is using the functor Hom (-,-) and the second is using element. It is essential in this work that the new formulation is stronger than the old ones.

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We introduce a definition of a "relatively divisible" module for a general ring, namely, a module A such that any element a A is divisible by the elements of the ring whose left annihilator is contained in the annihilator of a. This definition coincides with the definition of a divisible module in case that the ring is an integral domain.

A quotient of a relatively divisible module is a relatively divisible module iff the ring is a PP-ring.

It is known that a commutative integral domain is a Dedekind ring iff every divisible module is injective. We get similar results: A commutative integral domain is a Prüfer ring iff every divisible module is FP-injective. We get also similar results in case that the ring is not an integral domain: A commutative ring is a semi-hereditary ring iff it is a PP ring such that every relatively divisible module is FP-injective. A commutative ring is a Noetherian hereditary ring iff it is a PP ring such that every divisible module is injective.

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