ON PRÜFER RINGS AND REGULAR RINGS

BY

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Abstract

Characterizations of semihereditary rings, of PP-rings, of Prüfer rings, of Dedekind rings and of von Neumann regular rings are given. If gl.dimR=o then for every ordinal α $\mathrm{End}_R(R^{(\alpha)})$ is a von Neumann regular ring. A duality property between two types of exact sequences is given. FP-injectivity of modules is not preserved under localization. Over a domain which is not a division ring there exists an injective module which is not flat.

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0. Introduction. Throughout this paper, unless otherwise indicated, R will be a ring with a unit. All modules will be unitary left R-modules, and an ideal will be a left ideal. A regular ring will be regular in the sense of von-Neumann. A domain will be not necessarily commutative.

In section 1 we give characterizations of semihereditary rings, of PP-rings, of Prüfer rings, of noetherian rings and of Dedekind rings. If $\mathrm{gl.dim}R=0$ then for every ordinal α, $\mathrm{End}_R(R^{(\alpha)})$ is regular. We find duality between two types of exact sequences. A localization of FP-injective module does not have to be FP-injective.

In section 2 we give some characterizations of regular rings. As a result we get that a domain in which every injective module is flat must be a division ring.

The symbol (*) will always mean the exact sequence $0 \to A' \to A \to A'' \to 0$. For every left module M $\mathrm{Hom}_R(M,(*))$ will design the induced sequence $0 \to \mathrm{Hom}_R(M,A') \to \mathrm{Hom}_R(M,A) \to \mathrm{Hom}_R(M,A'') \to 0$. For every right module D, $D \otimes_R (*)$ will design the induced sequence $0 \to D \otimes_R A' \to D \otimes_R A \to D \otimes_R A'' \to 0$.

Definitions:

(i) The sequence (*) is called injective iff for every module M the the sequence $\mathrm{Hom}_R(M,(*))$ is exact.

(ii) The sequence (*) is called C-injective iff for every cyclic module M the sequence $\mathrm{Hom}_R(M, (*))$ is exact.

(iii) The sequence (*) is called CFP-injective iff for every cyclic finitely presented module M the sequence $\mathrm{Hom}_R(M, (*))$ is exact.

(iv) The sequence (*) is called pure iff for every right module D the sequence $D \otimes_R (*)$ is exact.

(v) The sequence (*) is called C-pure iff for every cyclic right module D the sequence $D \otimes_R (*)$ is exact.

(vi) The sequence (*) is called RD-pure iff for every $r \in R$ the sequence $R/rR \otimes_R (*)$ is exact.

D. Fieldhouse gave the following characterization for pure sequences ([4] theorem 2).

Proposition 0.1. The sequence (*) is pure iff for every finitely presented left module M $\mathrm{Hom}_R(M, (*))$ is exact.

Since every ideal is a direct limit of finitely generated ideals we have the sufficient condition for a sequence to be C-pure.

Propostion 0.2. The sequence (*) is C-pure if for every cyclic finitely presented module M, $\mathrm{Hom}_R(M, (*))$ is exact.

R.B. Warfield ([9] prop.2) characterized RD-pure sequences in a way similar to Fieldhouse's characterization of pure sequences.

Proposition 0.3. The following are equivalent:

(i) (*) is RD-pure ,

(ii) For every $r \in R$ $\mathrm{Hom}_R(R/Rr, (*))$ is exact.

(iii) For every $r \in R$ $rA' = A' \cap rA$, Theorem 0.4. Let α,β be ordinal numbers. Then $\mathrm{Hom}_R(R^{(\beta)}/G,(*))$ is exact for every submodule G with α generators of $R^{(\beta)}$ iff for every set of relations $\sum_{j=1}^{\alpha} r_{ij}x_j \varepsilon A'$ $i=1,\dots,\beta$ where $r_{ij}\varepsilon R$ are such that $r_{ij}=0$ for almost every j and where $x_j\varepsilon A$, there exist $y_j\varepsilon A'$ such that $\sum_j r_{ij}x_j=\sum_j r_{ij}y_j$.

D. Fieldhouse proved the theorem for the case where α,β are finite which is essentially the proof of prop. 0.1, and his proof can be extended trivially to the general case. Prop. 0.3 appears to be a special case of the theorem if we take $\alpha=\beta=1$.

Propostion 0.5. Consider the following cases :

(i) (*) is injective ,

(ii) (*) is C-injective .

(iii) (*) is CFP-injective .

(iv) (*) is pure .

(v) (*) is C-pure ,

(vi) (*) is RD-pure ,

Then: (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii)

(i) $\Longrightarrow$ (iv) $\Longrightarrow$ (iii) $\Longrightarrow$ (vi)

(iv) $\Longrightarrow$ (v) $\Longrightarrow$ (vi) .

Definitions: Let A be a left module.

(i) If for every finitely presented module M $\mathrm{Ext}^1(M,A)=0$ then A is called FP-injective (B. Stenström [8]).

(ii) If for every cyclic finitely presented module M $\mathrm{Ext}^1(M,A)=0$ then A is called CFP-injective.

(iii) If for every $r\in R$ $\mathrm{Ext}^1(R/Rr,A)=0$ then A is called RD-injective.

Propostion 0.6. Consider the following cases:

(i) A is injective .

(ii) A is FP-injective.

(iii) A is CFP-injective.

(iv) A is RD-injective ,

(v) A is divisible by regular elements.

Then (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (v).

Remark. It is a special case of a lemma of B.Stenström ([8] Lemma 3.1) that in coherent rings (iii) $\Longleftrightarrow$ (ii) .

1. Characterizations of rings by exact sequences

Let $\{A_\alpha\}$ be a set of left modules. Then we have

Lemma 1.1. The following are equivalent:

(i) Each A_α is FP-injective .

(ii) πA_α is FP-injective .

(iii) $\oplus A_\alpha$ is FP-injective.

Proof: (i) $\Longrightarrow$ (ii) Using the isomorphism $\mathrm{Hom}_R(D,\pi A_\alpha)\cong\pi\,\mathrm{Hom}_R(D,A_\alpha)$.

(ii) $\Longrightarrow$ (iii) we wish to show that the map $\mathrm{Hom}_R(R^n,\oplus A_\alpha)\longrightarrow\mathrm{Hom}_R(G,\oplus A_\alpha)$ is epimorphism whenever G is a finitely generated submodule of R^n. Take $f:G\longrightarrow\oplus A_\alpha$. Then we get the map $G\longrightarrow\oplus A_\alpha\longrightarrow\pi A_\alpha$ which can be extended by the assumption to a map $R^n\longrightarrow\pi A_\alpha$.

$$\begin{array}{ccccc} 0 & \longrightarrow & G & \longrightarrow & R^n \\ & & f\downarrow & & \downarrow \\ 0 & \longrightarrow & \oplus A_\alpha & \longrightarrow & \pi A_\alpha \end{array}$$

Since R^n is a finitely generated module, the image is in $\oplus A_\alpha$.

(iii) $\Longrightarrow$ (i) is obvious. Q.E.D.

Remark. We can prove similarly the lemma for CFP-injectivity (RD-injectivity) instead of FP-injectivity.

It seems from the following proposition that the notion of FP-injectivity generalizes the notion of injectivity in a way dual to the way that finitely generated projective modules specialize the projective modules.

Proposition 1.2. The ring R is semihereditary (a PP-ring) iff every quotient of an injective module is FP-injective (RD-injective)

Proof. Suppose that R is semihereditary and let A be a submodule of E, where E is a FP-injective module. Let G be a finitely generated submodule of R^n. Then we have the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{Hom}_R(R^n,E) & \longrightarrow & \mathrm{Hom}_R(R^n,E/A) & & \\ \downarrow & & \downarrow & & \\ \mathrm{Hom}_R(G,E) & \longrightarrow & \mathrm{Hom}_R(G,E/A) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ 0 & & 0 & & \end{array}$$

The second row is exact since G is finitely generated and hence projective by the assumption. The first column is exact since E is FP-injective. Therefore the second column is also exact, which shows that E/A is FP-injective

Conversely, suppose that every quotient of an injective module is FP-injective Let G be a finitely generated submodule of R^n. We shall show that G is project

Let $0 \longrightarrow A \longrightarrow E \longrightarrow E/A \longrightarrow 0$ be an exact sequence where E is an injective module, and consider the commutative diagram:

$$\begin{array}{ccccc} \mathrm{Hom}_R(R^n,E) & \longrightarrow & \mathrm{Hom}_R(R^n,E/A) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ \mathrm{Hom}_R(G,E) & \longrightarrow & \mathrm{Hom}_R(R,E/A) & \longrightarrow & 0 \\ & & \downarrow & & \\ & & 0 & & \end{array}$$

The second column is exact by the assumption and so the second row is also exact. Therefore G is projective. The proof of the case in brackets is similar. Q.E.D.

Corollary 1.3. If every module which is divisible by regular elements is FP-injective (RD-injective) then the ring is semihereditary (a PP-ring).

It follows from P.M. Cohn's theorem ([3] 2.4) that the sequence (*) is C-pure iff for every $\sum_i r_i a_i \varepsilon A'$ where $r_i \varepsilon R, a_i \varepsilon A$ there exist $b_i \varepsilon A'$ such that $\sum_i r_i b_i = \sum_i r_i a_i$. Using this criterion we get:

Lemma 1.4. Let R be an integral domain and let $0 \longrightarrow A \longrightarrow E \longrightarrow E/A \longrightarrow$ be an exact sequence where E is a divisible module. Then A is a divisible module iff the sequence is C-pure.

Proof. Let A be a divisible module, and let $\sum_i r_i e_i \varepsilon A$; $r_i \varepsilon R$; $e_i \varepsilon E$. Assume that $r_1 \neq 0$. Since A is divisible we have $a_1 \varepsilon A$ such that $r_1 a_1 = \sum r_i e_i$ and we can choose $a_r = \ldots = a_n = 0$.

Assume now that the sequence is C-pure. Then clearly it is RD-pure. Let $a \varepsilon A$ and $0 \neq r \varepsilon R$. Then $a \varepsilon E$ and since E is a divisible module there exists $e \varepsilon E$ such that $re = a$. But then from prop. 0.3 we have $b \varepsilon A$ such that $a = rb$. Q.E.D.

Lemma 1.5. If $f: A \longrightarrow B$ is a R-homomorphism such that for every divisible abelian group Q the sequence $0 \longrightarrow \mathrm{Hom}_Z(B,Q) \xrightarrow{f^*} \mathrm{Hom}_Z(A,Q)$ is exact, then f is an epimorphism

<u>Proof:</u> We have the commutative diagram

$$\begin{array}{ccccccccc}
 & & A & = & A & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & Im(f) & \longrightarrow & B & \longrightarrow & B/Im(f) & \longrightarrow & 0 \\
 & & \downarrow & & & & & &
\end{array}$$

which for every Q induces the following commutative diagram:

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
0 & \longrightarrow & Hom_Z(B/Im(f),Q) & \longrightarrow & Hom_Z(B,Q) & \longrightarrow & Hom_Z(Im(f),Q) \\
 & & & & \downarrow & & \downarrow \\
 & & 0 & \longrightarrow & Hom_Z(A,Q) & = & Hom_Z(A,Q)
\end{array}$$

This forces $Hom_Z(B/Im(f),Q)$ to be equal 0. But this implies that $B/Im(f)=0$, for otherwise if we take Q to be the injective hull of $B/Im(f)$ we have the insertion map in $Hom_Z(B/Im(f),Q)$ which is different from 0.

Q.E.D.

The following theorem gives duality property between the notion of CFP-injectivity of a sequence and that of C-purity.

<u>Theorem 1.6.</u>

(i) (*) is CFP-injective iff for every divisible group Q $Hom_Z((*),Q)$ is C-pure.

(ii) (*) is C-pure iff for every divisible group Q $Hom_Z((*),Q)$ is CFP-injective.

(iii) (*) is pure iff for every divisible group Q $Hom_Z((*),Q)$ is pure.

(iv) (*) is RD-pure iff for every divisible group Q $Hom_Z((*),Q)$ is RD-pure.

<u>Proof.</u>

(i) Use the isomorphism $Hom_Z(Hom_R(R/I,N),Q) \approx Hom_Z(N,Q) \otimes_R R/I$ where R/I is finitely presented and Q injective, (Bourbaki [1] pp.63 ex.14). In order to prove the "if" direction use Lemma 1.5.

(ii)-(iv) Use the isomorphism $Hom_R(R/I, Hom_Z(N,Q)) \approx Hom_Z(R/I \otimes_R N,Q)$ (Cartan-Eilenberg [2] Prop.II 5.2).

We give now two sufficient conditions for a domain to be a "noncommutative Prüfer ring" (i.e. a semihereditary domain) which turns out to be also necessary when the domain is commutative.

Proposition 1.7. Let R be a domain.

(i) If every C-pure sequence of left modules is pure then R is left semihereditary.

(ii) If every CFP-injective sequence of right modules is pure then R is left semihereditary.

Proof.

(i) Let A be a divisible module and let E be an injective module containing A. By Lemma 1.4 the sequence $0 \longrightarrow A \longrightarrow E \longrightarrow E/A \longrightarrow 0$ is C-pure and hence is pure by the assumption. Therefore A is FP-injective. The proposition follows from corollary 1.3 ,

(ii) By theorem 1.6 Q.E.D.

R.B. Warfield proved that in a Prüfer ring every RD-pure exact sequence in pure ([9] corollary 5). As a corollary of the proposition we get that the converse is also true. Moreover, to show this direction we do not need the commutativity of R.

Corollary 1.8. Let R be a domain such that every RD-pure exact sequence of left modules is pure. Then R is both left and right semihereditary.

We wish to have a similar characterization of Dedekind rings. To this end we show:

Proposition 1.9. A ring is noetherian iff every CFP-injective sequence is C-injective.

Proof. If the ring is noetherian the proposition is clear. Suppose that every CFP-injective sequence is C-injective. Then every CFP-injective module is injective. Therefore from B.H. Maddox ([6] pp.157) R is noetherian. Q.E.D.

Since a Dedekind ring is both Prüfer and noetherian we have:

Corollary 1.10. Let R be a commutative domain. Then R is a Dedekind ring iff every RD-pure sequence is C-injective.

Remark. A domain is a P.I.D. iff for every ideal I, R/I=0. Therefore it is trivial that a domain is a P.I.D. iff every exact sequence is C-injective.

Propostion 1.11. Let R be a domain. Then there is no nontrivial ideal in R which is RD-pure in R.

Proof. Suppose that I is RD-pure in R. Then for every divisible abelian group Q the sequence $0 \longrightarrow \mathrm{Hom}_z(R/I,Q) \longrightarrow \mathrm{Hom}_z(R,Q) \longrightarrow \mathrm{Hom}_z(I,Q) \longrightarrow 0$ is RD-pure (theorem 1.6 (iv)). But $\mathrm{Hom}_z(R,Q)$ is injective. Therefore from lemma 1.4 we get that the sequence is C-pure and again from theorem 1.6 the sequence $0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$ is CFP-injective. But this implies that I is a direct summand in R, which forces I to be trivial, since R is a domain. Q.E.D.

Let R be a commutative ring. For any multiplicative set S denote by R_s the localization of R by S.

It is quite known that a sequence is pure iff every localization by maximal ideal is pure. We shall show that CFP-injectivity of a sequence is preserved under localization and vice versa.

Proposition 1.12. If an exact sequence is CFP-injective then every localization of the sequence is R_S-CFP-injective, and hence also R-CFP-injective.

Proof. Let $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ be a CFP-injective sequence of R modules. Let I be a finitely generated ideal in R_S with generators $\frac{r_1}{s_1},\ldots,\frac{r_k}{s_k}$. Then $J=(r_1,\ldots,r_k)$ is an ideal in R such that $J_S=I$. By the assumption the sequence

$0 \longrightarrow Hom_R(R/J,A') \longrightarrow Hom_R(R/J,A) \longrightarrow Hom_R(R/J,A'') \longrightarrow 0$ is exact and hence its localization. But since J is finitely generated we know that $Hom_R(R/J,B)_S=Hom_{R_S}((R/J)_S,B_S)$. We know also that $(R/J)_S=R_S/J_S$. But $J_S=I$. So the sequence $0 \longrightarrow Hom_{R_S}(R_S/I,A') \longrightarrow Hom_{R_S}(R_S/I,A) \longrightarrow Hom_{R_S}(R_S/I,A'')$-- is exact, as required.

Using the isomorphism $Hom_R(C,A_S)_S=Hom_{R_S}(C_S,A_S)$ we get the second part of the proposition, Q.E.D.

Proposition 1.13. If for every maximal ideal m the localized sequence is R_m-CFP-injective, then the sequence is R-CFP-injective.

Proof. Let $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ be an exact sequence satisfying the hypothesis. Let I be a finitely generated ideal in R. Then I_S is a finitely generated ideal in R_S and $(R/I)_S=R_S/I_S$. Let

$0 \longrightarrow Hom_R(R/I,A') \longrightarrow Hom_R(R/I,A) \longrightarrow Hom_R(R/I,A'') \longrightarrow C \longrightarrow 0$

be an exact sequence. Then for every maximal ideal m the sequence

$0 \longrightarrow Hom_R(R/I,A')_m \longrightarrow Hom_R(R/I,A)_m \longrightarrow Hom_R(R/I,A'')_m \longrightarrow C_m \longrightarrow 0$

is exact. Using the isomorphism $Hom_R(R/I,B)_m=Hom_{R_m}((R/I)_m,B_m)$ we get that the sequence

$$0 \longrightarrow Hom_{R_m}(R_m/I_m, A'_m)_m \longrightarrow Hom_{R_m}(R_m/I_m, A_m) \longrightarrow Hom_{R_m}(R_m/I_m, A''_m) \longrightarrow C_m \longrightarrow 0$$

is exact and from the assumption it follows that every $C_m=0$. Therefore $C=0$

Q.E.D.

Remark: Proposition 1.12, 1.13 can be stated and proved similarly for FP-injectivity and RD-purity.

Corollary 1.14. There exists a ring R and an R-module A which is FP-injective and a maximal ideal m such that A is not R_m- FP-injective.

Proof: B. Stenstrom proved that a ring is coherent iff the direct limit of FP-injective module is FP-injective. M.E.Harris ([5] theorem 3) gave an example of a non coherent ring R such that for every maximal ideal m R_m is a coherent ring.

The example of Harris satisfies the condition of the corollary Q.E.D.

Q.E.D.

Proposition 1.15. If $gl.dim_{\bullet} R=0$ then $End_R(R^{(\alpha)})$ is a regular ring for every ordinal α.

Proof: An element of $End_R(R^{(\alpha)})$ is an infinite matrix with a finite number of elements different from 0 in every row. Let $\bar{r}=(r_{ij})$ be such an element and let (a_j) be an arbitrary element of $R^{(\alpha)}$. Consider the ideal in R $I=(\sum_j r_{ij}a_j)_i$. Since $gl.dim_{\bullet} R=0$ the sequence $0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$ is injective, which means that there exist $y_k \varepsilon I$, hence of the form $y_k=\sum_\ell t_{k\ell}(\sum_j r_{\ell j}a_j)$, such that $\sum_j r_{ij}y_j=\sum_j r_{ij}a_j$. Substitution of y_k gives $\sum_{k,\ell,j} r_{ik}t_{k\ell}r_{\ell j}a_j=\sum_j r_{ij}a_j$. If we put now $a_j=\delta_{jp}$ we finally get $\sum r_{ik}t_{k\ell}r_{\ell p}=r_{ip}$. Therefore we found an element $\bar{t}=(t_{ij})\varepsilon End_R(R^{(\alpha)})$ such that $\bar{r}\bar{t}\bar{r}=\bar{r}$. Q.E.D

Remark: Using the same argument we can prove the well known result: The ring of matrices over a regular ring is regular.

2. Characterizations of von Neumann regular rings.

Lemma 2.1. Let R be a ring CFP-injective over itself. Then every finitely generated projective ideal of R is generated by an idempotent

Proof. Let I be a finitely generated projective ideal of R. Then there are $a_\alpha \in I$ and $\phi_\alpha : I \longrightarrow R$, finite in number, such that for every $a \in I$ $a = \sum_\alpha \phi_\alpha(a) a_\alpha$ Since R is CFP-injective over itself each ϕ_α can be extended to a homomorphism $\psi_\alpha : R \longrightarrow R$. Therefore $a = \sum_\alpha a\psi_\alpha(1) a_\alpha$. We can take a out of the sum since the set $\{a_\alpha\}$ is finite. Therefore $\sum_\alpha \psi_\alpha(1) a_\alpha$ is the idempotent generating I. Q.E.D.

Remark: If R is a domain CFP-injective over itself we can use the same argument to get that every projective ideal is generated by an idempotent. But this is impossible in a domain unless the ideal is trivial. Therefore in a domain CFP-injective over itself there are no projective ideals.

An immediate corollary of the lemma is a theorem of E.R.Gentile (7) A semihereditary ring CFP-injective over itself is regular. This theorem appears to be a special case of Proposition 2.3, since if R is semihereditary then it is coherent and $w.\dim R < \infty$

Proposition 2.2. A ring R is regular if every principal ideal is PD-pure in R.

Proof. Let $r \in R$ and consider the ideal Rr. By the assumption Rr is PD-pure in R, hence taking $r \in R$ and $1 \in R$ we have by Prop.0.3 $x \in Rr$ such that $rx = r1$. But x is of the form $x = sr$, which shows that $rsr = r$. Q.E.D.

Proposition 2.3. Let R be a ring such that $w.\dim R < \infty$ and every injective right module is flat. Then R is regular.

Proof. Let $0 \longrightarrow A_n \longrightarrow P_n \longrightarrow A_n \longrightarrow 0$ be an exact sequence where P_n is a projective module. We show that if A_n is flat then also A_{n-1} is.

Let D be a right module and let E be an injective module containing D. Consider the commutative diagram:

$$\begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ 0 & \dashrightarrow & D \otimes A_n & \longrightarrow & D \otimes P_n \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & E \otimes A_n & \longrightarrow & E \otimes P_n \end{array}$$

We get that the sequence is pure and hence A_{n-1} is flat. Q.E.D.

<u>Proposition 2.4.</u> Let R be a domain. Then either there exists an injective module which is not flat or R is a division ring.

<u>Proof.</u> Suppose every injective module is flat. We first prove that R is semihereditary. In order to see this we show that every divisible module is FP-injective (corollary 1.3). Let A be a divisible module and let $0 \longrightarrow A \longrightarrow E \longrightarrow E/A \longrightarrow 0$ be an exact sequence where E is injective. Since R is a domain the sequence is C-pure (lemma 1.4) and since E is also flat the sequence is pure. Therefore A is FP-injective. Thus we saw that R is semihereditary. From Prop.2.3 R is regular. Since R is a domain it must be a division ring. Q.E.D.

B. Stenström had proved that if R is coherent, FP-injective over itself and w.dim $R < \infty$ then R is regular.

Also he had shown that if R is coherent and FP-injective over itself then every flat module is FP-injective.

Therefore the following is a generalization of his result.

<u>Proposition 2.5.</u> If every flat R module is FP-injective and if w.dim$R < \infty$ then R is regular.

<u>Proof.</u> Let $0 \longrightarrow A_n \longrightarrow P_n \longrightarrow A_{n-1} \longrightarrow 0$ be an exact sequence where P_n is projective. We show that if A_n is flat then also A_{n-1} is. To this end we prove that the sequence is pure.

Let $0 \longrightarrow G \longrightarrow R^k \longrightarrow M \longrightarrow 0$ be a finite presentation. Consider the commutative diagram:

$$\begin{array}{ccccccccc}
 & & \mathrm{Hom}_R(M,A_n) & \longrightarrow & \mathrm{Hom}_R(M,P_n) & \longrightarrow & \mathrm{Hom}_R(M,A_{n-1}) & \dashrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathrm{Hom}_R(R^k,A_n) & \longrightarrow & \mathrm{Hom}_R(R^k,P_n) & \longrightarrow & \mathrm{Hom}_R(R^k,A_{n-1}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_R(G,A_n) & \longrightarrow & \mathrm{Hom}_R(G,P_n) & \longrightarrow & \mathrm{Hom}_R(G,A_{n-1}) & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

By the "Snake lemma" the first row is exact and hence the sequence is pure (Prop.0.1)

Q.E.D.

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